RESPONSES TO QUESTIONS ON FROBENIOIDS

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The Geometry of Frobenioids I: The General Theory ([FrdI])

Question 1: Definition 1.2, (i): Since the validity of the condition that two arrows be "metrically equivalent" depends solely on the images via "Div" of the two arrows, might it not be better to call such arrows "Div-equivalent" and then, to avoid confusion, to use the term "Φ-equivalent" for arrows that, in the current terminology, are called "Div-equivalent"?
Response 1:
First of all, the terminology "Φ-equivalent" is not very good since it depends on the notation "Φ" for a *specific* monoid on a base category. The sense underlying the terminology *"base-equivalent"/ "Div-equivalent"* is the idea that the pair of morphisms induce the *same* morphism on the base object/collection of divisors of the base object. That is to say, these two terms (i.e., base-/Div-equivalent) are most naturally regarded as being "parallel". Thus, the term *"divisor-equivalent"* would be one reasonable alternative to "Div-equivalent". By contrast, the sense underlying the terminology *"metrically equivalent"/"isometric"* is that it corresponds, in the case of Frobenioids that arise from *metrized line bundles*, to the idea that the morphism(s) involved induce(s) the *same/no* discrepancy in metrics. That is to say, from the point of view of maintaining the close relationship to the term *"isometric"*, it does not seem to me that it would be desirable to replace "metrically equivalent" by "Div-equivalent".

Question 2: Definition 1.2, (ii): Are there any other equivalent definitions of the notion of a pull-back morphism?
Response 2:
I am not aware of any natural equivalent definitions. It seems that the definition given is sufficiently natural, relative to the way in which the term "pull-back morphism" is typically applied in arithmetic geometry.

Question 3: Definition 1.2, (iii): Is it true that, if one restricts one's attention to the portion of the theory of Frobenioids that is used in the IUTeich papers, then the only Frobenioids in which there exist morphisms that are not co-angular are the archimedean Frobenioids?
Response 3:
Yes. Indeed, the only Frobenioids that are used in the IUTeich papers are the following:

(F1) *tempered Frobenioids*, i.e., a generalization developed in [EtTh], §3, §4, §5, of the geometric example given in [FrdI], Example 6.1, to the case of the *tempered coverings* that arise in the theory of the theta function;

(F2) *Frobenioids* associated to *number fields* as in [FrdI], Example 6.3 (cf., e.g., [IUTchIII], Example 3.6), together with the *realifications* associated to (certain of) such Frobenioids;

(F3) certain special cases of the *p-adic Frobenioids* discussed in [FrdII], Example 1.1 (cf., e.g., the Frobenioids "$\mathcal{C}_{\underline{v}}$", "$\mathcal{C}_{\underline{v}}^{\vdash}$" considered in [IUTchI], Example 3.3, (i), which also appear in R11 and R13 — i.e., the responses to questions Q11 and Q13 — below), together with various related Frobenioids obtained by forming associated *realifications* or by forming the quotient "$\mathcal{O}^{\times\boldsymbol{\mu}} = \mathcal{O}^{\times}/\mathcal{O}^{\boldsymbol{\mu}}$" of the units "$\mathcal{O}^{\times}$" by the torsion subgroup "$\mathcal{O}^{\boldsymbol{\mu}} \subseteq \mathcal{O}^{\times}$";

(F4) copies of the *archimedean Frobenioid* discussed in [FrdII], Example 3.3, (ii) (i.e., the Frobenioid denoted "$\mathcal{C}$").

Here, we note that (F3) and (F4) are *inessential* since a Frobenioid as in (F3) essentially amounts to (i.e., may be replaced by) a suitable topological monoid with a continuous action by a topological group, while a Frobenioid as in (F4) essentially amounts to a copy of the topological monoid $\mathcal{O}_{\mathbb{C}}^{\triangleright}$ (i.e., the multiplicative topological monoid of nonzero complex numbers of absolute value ≤ 1). In this context, it should, however, be pointed out that, in fact, (F3) is a bit *"less inessential"* than (F4) since (F3) occurs in the theory of (F1). At any rate, from the point of view of studying IUTeich,

(I1) in [FrdI], one may assume that all Frobenioids are *model Frobenioids* (cf. [FrdI], Theorem 5.2, (ii)), which implies, in particular, that every object of a Frobenioid is *isotropic*, and that every morphism of a Frobenioid is *co-angular*;

(I2) one may in fact *ignore* [FrdII], §3, §4, §5.

Here, we note that (I1) results in a *substantial simplification* of the theory of [FrdI].

<u>Question 4</u>: Definition 1.2 and the Chart of Types of Morphisms in a Frobenioid at the end of [FrdI]: The definitions of many types of morphisms concern properties of 1) the projection to the base category, 2) the zero divisor image, 3) the Frobenius degree. Might it not be easier to keep track of the terms (and, moreover, allow one to simplify the Chart of Types of Morphisms) if one instead refers to linear morphisms as 3-morphisms, to isometries as 2-morphisms, to base-isomorphisms as 1-morphisms, to pre-steps as 13-morphisms, to morphisms of Frobenius type (in light of later results) as co-angular 12-morphisms, to pull-back morphisms (in light of later results) as co-angular 23-morphisms, and to isometric pre-steps as 123-morphisms? Then co-angular 123-morphisms are isomorphisms, etc.
<u>Response 4</u>:
The point of the naming conventions introduced in Definition 1.2 was precisely to give *"combinatorial"*, or *"mechanical"* (i.e., involving "1", "2", and "3") characterizations of various types of morphisms that typically appear in arithmetic geometry. That is to say, the terminology was chosen so as to *suggest well-known*

situations familiar from conventional arithmetic geometry. The content of the definitions of this terminology was then intended to be a sort of *translation* of terminology that was close to familiar wording in conventional arithmetic geometry *into* combinatorial/mechanical/"123"-type language. If, however, you have a proposal for a slightly different sort of table involving "123", then you are certainly welcome to construct such a table.

Question 5: Definition 1.3 contains 7 parts, some of which consist of subparts. Might it be possible, if one restricts one's attention to the applications to IUTeich, to reduce the number of defining properties?
Response 5:
In light of (I1) of R3, properties (iii), (a), (b); (v); and (vii) of Definition 1.3 may be omitted.

Question 6: Remark 3.1.2: "the image of $1 \in \mathbb{Z}_{\geq 0}$": Does this expression in fact refer to "the image of $(1, 1) \in \mathbb{F}$"?
Response 6:
Yes. I believe that the intended meaning is clear from the context, but this issue is addressed in the newly released version of the Comments for [FrdI].

Question 7: Theorem 5.2, (i), (b): Should "to as the projection to $\mathcal{D}$ to ϕ" be replaced by "to as the projection to $\mathcal{D}$ of ϕ?
Response 7:
Thank you. This is indeed a misprint which is corrected in the newly released version of the Comments for [FrdI].

Question 8: I have not read the IUTeich papers yet. I wonder if there are Theorems and Propositions of [FrdI] whose statements hold by obvious reasons for all the Frobenioids that appear in the papers on IUTeich (and which therefore may be omitted in one's preparation for studying the IUTeich papers)?
Response 8:
As discussed in R3, once one *assumes* (I1), the entire theory of [FrdI] become *much easier*. One approach (cf. R3) to presenting the theory of Frobenioids to people whose main interest is to prepare for studying the IUTeich papers might be

(A1) to start by discussing the *fundamental example* of [FrdI], Definition 1.1, (iii);

(A2) then proceed to discussing the explicit construction of *model Frobenioids* as in [FrdI], Theorem 5.2 (cf. also Proposition T in R23 below);

(A3) explain how, by specifying appropriate *divisor monoids* and *rational function monoids*, one may obtain [FrdI], Examples 6.1, 6.3; [FrdII], Example 1.1, as special cases of the notion of a *model Frobenioid*;

(A4) then proceed to discussing the [FrdI], Corollary 4.11 (which, in fact, may, from the point of view of studying IUTeich, simply be *accepted on faith*!), in the case of *model Frobenioids*, explaining a few of the main, representative ideas of the proofs (such as [FrdI], Remark 3.1.2).

In this context, I should perhaps mention that, other than [FrdI], Corollary 4.11, perhaps the most important property of the theory of Frobenioids that is used in [EtTh], §3, §4, §5 (and hence also in IUTeich) is [FrdI], Proposition 5.6.

Question 9: Remark 6.3.1: How much is known about the use of Frobenioids to study arithmetic line bundles on regular proper models of curves over number fields? Should one look at Frobenioids whose base category is a Frobenioid? Or should one perhaps develop a theory of two-dimensional Frobenioids, where the monoid of positive integers is replaced by the monoid of positive elements of $\mathbb{Z} \times \mathbb{Z}$ equipped with the lexicographical ordering?
Response 9:
Note that in Example 6.1, there is *no restriction* on the *dimension* of the variety V. In particular, the nontriviality of extending Example 6.3 to the case of arithmetic line bundles on (two-dimensional) regular proper models of curves over number fields does *not* lie in the *two-dimensionality* of such models. In particular, there is *no issue* of being having to contend with "two-dimensional Frobenioids" or monoids more complicated than $\mathbb{N}$ that arise from lexicographic orderings. Rather, the nontriviality (i.e., the "complications" referred to in Remark 6.3.1) of extending Example 6.3 as discussed in Remark 6.3.1 lies in the issue of how to represent, in a *category-theoretic fashion*, the notion of an *arbitrary (say, $\mathcal{C}^\infty$-class) metric* on the archimedean localization of an arithmetic line bundle. I have not thought about this issue in much detail, but it appears to involve various nontrivial technical complications. On the other hand, another possibility for further development of the theory of Frobenioids that is not particularly related to archimedean localizations, but is related to more complicated monoids is the following:

> One could try to construct some sort of more general type of Frobenioid that corresponds to the *"tautological system of monoids corresponding to valuations on Berkovich spaces"*.

That is to say, since each point of a Berkovich space corresponds to a valuation, each such point has a natural monoid (i.e., the monoid determined by the valuation) attached to it. One could then try to consider the abstract topological space (i.e., determined by a Berkovich space) equipped with this tautological system of monoids, construct some sort of associated category (i.e., in the spirit of the notion of a Frobenioid), and then see to what extent various properties of this category can be reconstructed *category-theoretically*, i.e., without using, for instance, some given p-adic scheme that gave rise to the Berkovich space.

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Question 10: Example 1.1, (i): The object $\Phi_0^{\mathbb{Z}}$ is not defined in [FrdII]. As a result, in the first displayed line of (ii), Φ_0^{Λ} is not defined for the type $\Lambda = \mathbb{Z}$, i.e., there is no explicitly stated restriction there to the effect that $\Lambda \neq \mathbb{Z}$.
Response 10:
Thank you. This is indeed a slight oversight in the use of notation which is addressed in the newly released version of the Comments for [FrdII].

Question 11: Example 1.1, (ii): In the third line of this example, is it assumed that the output monoprime monoids of Φ are of type Λ?
Response 11:
No, this is certainly not assumed. For instance, in the case of the p-adic Frobenioid "$\mathcal{C}_{\underline{v}}$" considered in [IUTchI], Example 3.3, (i), the output monoprime monoids of Φ are of type $\mathbb{Q} \neq \Lambda = \mathbb{Z}$.

Question 12: Example 1.1, (ii): What is $\Phi(K)$? Probably "K" is used as a shorthand for "$\mathrm{Spec}(K)$", an object of $\mathcal{D}_0$, but even then Φ is a functor defined on $\mathcal{D}$, not on $\mathcal{D}_0$.
Response 12:
Thank you. This is indeed a misprint which is corrected in the newly released version of the Comments for [FrdII].

Question 13: Example 1.1, (ii): I do not understand the meaning of "absolutely primitive" and "fieldwise saturated". (Here, I took into account the correction in the Comments for [FrdII], replacing $\Phi(K)$ by $\Phi(K)^{\mathrm{gp}}$ in the definition of "fieldwise saturated".) The current definitions appear to imply the following conclusions: if $\Lambda = \mathbb{Z}$ (resp. $\mathbb{Q}$, resp. $\mathbb{R}$), then Φ is absolutely primitive iff the output monoprime monoids of Φ are of type $\mathbb{Z}$ (resp. $\mathbb{Z}$ or $\mathbb{Q}$, resp. $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$); Φ is always fieldwise saturated. Is this what was intended?
Response 13:
First of all, it is true that Φ is *always absolutely primitive* if $\Lambda = \mathbb{R}$. Also, you are correct in asserting that, when $\Lambda = \mathbb{Q}$, Φ is *absolutely primitive* if and only if the output monoprime monoids of Φ are of type $\leq \Lambda = \mathbb{Q}$ (i.e., relative to the ordering on $\{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ induced by the relation of inclusion). In particular, *the notion of being absolutely primitive is only interesting when* $\Lambda = \mathbb{Z}$. On the other hand, I was entirely unable to understand the remaining cases of your assertion. Write $\mathrm{ord}(p) \in \mathrm{ord}(\mathbb{Q}_p^\times)$ for the element determined by $p \in \mathbb{Q}_p$. Let $n \in \mathbb{N}_{\geq 1}$. Consider the case where $\mathcal{D} = \mathcal{D}_0$, and Φ is given by the assignment

$$\mathrm{Spec}(K) \mapsto \tfrac{1}{n} \cdot \mathbb{N} \cdot \mathrm{ord}(p)$$

(for $\mathrm{Spec}(K) \in \mathrm{Ob}(\mathcal{D}_0)$). (Thus, the output monoprime monoids of Φ are of type $\mathbb{Z}$.) Then Φ is *not fieldwise saturated for any* $n \in \mathbb{N}_{\geq 1}$. (Indeed, if the ramification index e_K of K over $\mathbb{Q}_p$ does *not* divide n, then $\mathrm{ord}(K^\times) = \frac{1}{e_K} \cdot \mathbb{Z} \cdot \mathrm{ord}(p) \not\subseteq \frac{1}{n} \cdot \mathbb{Z} \cdot \mathrm{ord}(p) = \Phi(\mathrm{Spec}(K))^{\mathrm{gp}}$.) Moreover, when $\Lambda = \mathbb{Z}$, it holds that Φ is *absolutely primitive* if and only if $n = 1$. (Indeed, $\Phi(\mathrm{Spec}(K)) = \frac{1}{n} \cdot \mathbb{N} \cdot \mathrm{ord}(p) \subseteq \mathbb{Z} \cdot \mathrm{ord}(p) = \Lambda \cdot \mathrm{ord}(\mathbb{Q}_p^\times)$ if and only if $n = 1$.) Finally, I should perhaps mention that this sort of Φ (for $\Lambda = \mathbb{Z}$) in the case $n = 1$ appears in the case of the p-adic Frobenioid "$\mathcal{C}_{\underline{v}}^{\vdash}$" considered in [IUTchI], Example 3.3, (i).

Question 14: Remark 1.2.2, the last sentence: If one works with idèles, it is not unnatural to consider situations where p is mapped to p times a unit.
Response 14:
Your assertion does not appear to contradict the assertion that the situation considered in Remark 1.2.2 "*never arises in conventional scheme theory*". That is to say, "conventional scheme theory" refers to the situation where one only considers morphisms of schemes, i.e., (when the schemes involved are *affine*, such as $\mathrm{Spec}(\mathbb{Q}_p)$) *ring homomorphisms*. The point here is that it is *never* the case (at least when $u \neq 1$) that an assignment $\mathbb{Q}_p^\times \ni p \mapsto p \cdot u \in \mathbb{Q}_p^\times$ of the sort considered here arises from a *ring homomorphism* $\mathbb{Q}_p \to \mathbb{Q}_p$.

Question 15: Example 1.3, (i): What is $\mathcal{B}^{\mathrm{temp}}(\Pi)_{\Pi/\Pi_0}$, i.e., what is the role of the subindex Π/Π_0?
Response 15:
This is a special case of the notational convention "$\mathcal{C}_A$" (i.e., where A is an object of the category $\mathcal{C}$) discussed at the beginning of the discussion of "Categories" in [FrdI], §0.

Question 16: Definition 2.2, (i), and Example 2.2, (ii), (b), "$A_{\mathcal{D}}$ is Galois": What is the definition of a Galois object? I was unable to find the definition of a Galois object in [FrdI], [FrdII].
Response 16:
Here, the category "$\mathcal{D}$" under consideration is, as discussed in Example 1.3, (i), a subcategory of a category (i.e., *"temperoid"*) of the sort discussed in [SemiAnbd], §3 (cf. [SemiAnbd], Definition 3.1, (ii)). The term "Galois" is defined in [SemiAnbd], Definition 3.1, (iv).

Question 17: Definition 2.2, (i): H is a normal open subgroup of G, and, in particular, it is possible that $H = G$. Why then is the homomorphism $G \to G_A$ determined up to composition with an inner automorphism, while $H \to H_A$ is only well-defined up to composition with conjugation by an element of G?
Response 17:
The arrow $H \twoheadrightarrow H_A$ of the third display of Definition 2.2, (i), is determined precisely by *restricting* the arrow $G \twoheadrightarrow G_A$ of the second display of Definition 2.2, (i). Thus, the G-conjugacy indeterminacies of the latter arrow $G \twoheadrightarrow G_A$ *necessarily induce* G-conjugacy indeterminacies on the former arrow $H \twoheadrightarrow H_A$. There is no reason to expect that these indeterminacies may be "reduced" to H-conjugacy indeterminacies.

Question 18: Definition 2.2, (ii): I could not find any explicit definition of the action of H on $\boldsymbol{\mu}_N(A)$. Does it act via the homomorphism $H \to H_A$?
Response 18:
Yes. My understanding was that this is implicit in the phrasing "the natural surjective homomorphism $H \twoheadrightarrow H_A$ induces *isomorphisms*" that immediately precedes the first display of Definition 2.2, (ii).

Question 19: Definition 2.2, (iii): The wording here seems to imply that the notions of "fieldwise saturated" and "absolutely primitive" are complementary to one other. As I wrote in Q13, I was unable to understand the definitions of these terms.

Response 19:
The assertion that (in the situation under consideration) if Φ is absolutely primitive, then it is not fieldwise saturated follows immediately from an argument of the sort discussed in R13: that is to say, it suffices to consider the case where $e_K \neq 1$, so $\mathrm{ord}(K^\times) = \frac{1}{e_K} \cdot \mathbb{Z} \cdot \mathrm{ord}(p) \not\subseteq \Phi(\mathrm{Spec}(K))^{\mathrm{gp}} \subseteq \mathbb{Z} \cdot \mathrm{ord}(p) = \Lambda \cdot \mathrm{ord}(\mathbb{Q}_p^\times)$.

Question 20: Beginning of the proof of Theorem 2.4, the second sentence: could you provide a verification of this fact?
Response 20:
This fact follows immediately from the definitions (cf. the arguments used in R13 and R19).

Question 21: Definition 3.1, (v): Should $\mathrm{Aut}_{\mathcal{F}_A}(B)$ be replaced by $\mathrm{Aut}_{F_A}(B \to A)$?
Response 21:
This is not a misprint, but instead reflects my understanding that in this context (i.e., where one considers "B" as an object of "$\mathcal{F}_A$"), the "object of $\mathcal{F}_A$" denoted by "B" is precisely the object determined by the arrow $B \to A$ under consideration. Nevertheless, this issue is addressed in the newly released version of the Comments for [FrdII].

Question 22: Example 3.3: Is the category $\mathcal{C}_0$ equivalent to some model Frobenioid?
Response 22:
No. It is clear from the discussion of Example 3.3, (i), (ii) (cf. also Definition 3.1, (iii)) that $\mathcal{C}_0$ contains *non-isotropic objects*. Thus, the fact that $\mathcal{C}_0$ is not equivalent to a model Frobenioid follows formally from the fact (cf. Theorem 5.2, (ii)) that a model Frobenioid is necessarily of *isotropic type*.

Both [FrdI] and [FrdII]:

Question 23: [FrdI], Examples 6.1 and 6.3, "one verifies immediately that a morphism $\mathcal{L} \to \mathcal{M}$ ... may be thought of as consisting ...", and [FrdII], Example 1.1, "we observe in passing that an object of $\mathcal{C}_0$ lying over $\mathrm{Spec}(K)$ may be thought of as a metrized line bundle on $\mathrm{Spec}(K)$ ...": It seems that both are special cases of a general property of model Frobenioids that allows one to think of the objects of a model Frobenioid as metrized line bundles. If this is the case, then it seems that it would be preferable to provide, for ease of reference, an explicit statement of such a general property and its proof.
Response 23:
Both instances of this pattern are immediate from the definitions and simply amount to the well-known observation that a (metrized) line bundle may be understood as a rational equivalence class of divisors. It does not appear to be feasible to formulate this sort of pattern as a "general property" of Frobenioids since, in general, *only the "rational equivalence class of divisors"* aspect appears in the general theory of Frobenioids. That is to say, the *"(metrized) line bundle"* aspect does *not* appear in the general theory of Frobenioids; it only appears in specific situations where one *constructs* a Frobenioid as a suitably defined category of (metrized) line bundles on a(n) (arithmetic) scheme. In particular, since it appears difficult to formulate such constructions from scheme theory in any sort of generality, it does not appear realistic (at least relative to the theory of Frobenioids in its present form) to expect a formulation of a "general property" of the sort that you appear to have in mind. I suppose, however, that one *tautological* approach (i.e., as in [IUTchIII], Example 3.6, (i)) to making the construction of [FrdI], Theorem 5.2, (i), look more like a category of (metrized) line bundles is the following result (whose proof is immediate from the definitions):

Proposition T. (Tautological Torsor-theoretic Approach to Model Frobenioids) *Let $\mathcal{D}$, $\underline{\Phi}$, $\mathbb{B}$, $\mathrm{Div}_{\mathbb{B}} : \mathbb{B} \to \underline{\Phi}^{\mathrm{gp}}$, and $\underline{\mathcal{C}}$ be as in [FrdI], Theorem 5.2. Write $\underline{\mathcal{C}}^{\mathrm{tor}}$ for the category defined as follows:*

- *An* **object** *of $\underline{\mathcal{C}}^{\mathrm{tor}}$ is a triple*

$$(A_{\mathcal{D}}, T_A, \tau_A)$$

*where $A_{\mathcal{D}} \in \mathrm{Ob}(\mathcal{D})$; T_A is a $\mathbb{B}(A_{\mathcal{D}})$-**torsor**; τ_A is a* **trivialization** *of the $\underline{\Phi}(A_{\mathcal{D}})^{\mathrm{gp}}$-torsor obtained from T_A by executing the "change of structure group" operation determined by the homomorphism $\mathrm{Div}_{\mathbb{B}}(A_{\mathcal{D}}) : \mathbb{B}(A_{\mathcal{D}}) \to \underline{\Phi}(A_{\mathcal{D}})^{\mathrm{gp}}$. Thus, for any $d \in \mathbb{Z}$, we obtain an object*

$$(A_{\mathcal{D}}, T_A^{\otimes d}, \tau_A^{\otimes d})$$

by executing the "change of structure group" operation determined by the homomorphism $\mathbb{B}(A_{\mathcal{D}}) \to \mathbb{B}(A_{\mathcal{D}})$ given by multiplication by d.

- *a* **morphism** *of $\underline{\mathcal{C}}^{\mathrm{tor}}$*

$$\phi : A \stackrel{\mathrm{def}}{=} (A_{\mathcal{D}}, T_A, \tau_A) \;\to\; B \stackrel{\mathrm{def}}{=} (B_{\mathcal{D}}, T_B, \tau_B)$$

is a collection of data as follows: (a) an element $d \in \mathbb{N}_{\geq 1}$; (b) a morphism $\mathrm{Base}(\phi) : A_{\mathcal{D}} \to B_{\mathcal{D}}$, which determines [i.e., by executing the "change of *structure group" operations determined by the homomorphisms* $\mathbb{B}(B_{\mathcal{D}}) \to \mathbb{B}(A_{\mathcal{D}})$, $\underline{\Phi}(B_{\mathcal{D}})^{\mathrm{gp}} \to \underline{\Phi}(A_{\mathcal{D}})^{\mathrm{gp}}$*] an object* $\phi^* B = (A_{\mathcal{D}}, \phi^* T_B, \phi^* \tau_B)$*; (c) an* **isomorphism of** $\mathbb{B}(A_{\mathcal{D}})$**-torsors**

$$T_A^{\otimes d} \xrightarrow{\sim} \phi^* T_B$$

that maps the trivialization $\tau_A^{\otimes d}$ *to an element in the* $\underline{\Phi}(A_{\mathcal{D}})$*-orbit of the trivialization* $\phi^* \tau_B$*. Composites of morphisms are defined in the evident fashion.*

Then there is a **natural equivalence of categories**

$$\underline{\mathcal{C}} \xrightarrow{\sim} \underline{\mathcal{C}}^{\mathrm{tor}}$$

defined as follows: An object $(A_{\mathcal{D}}, \alpha)$ *of* $\underline{\mathcal{C}}$ *[cf. [FrdI], Theorem 5.2, (i)] is mapped to the object* $(A_{\mathcal{D}}, T_A, \tau_A)$ *of* $\underline{\mathcal{C}}^{\mathrm{tor}}$*, where* T_A *is the trivial* $\mathbb{B}(A_{\mathcal{D}})$*-torsor, and* τ_A *is the trivialization of* T_A *obtained by shifting the tautological trivialization of* T_A *by the element* $-\alpha \in \underline{\Phi}(A_{\mathcal{D}})^{\mathrm{gp}}$*. A morphism* $(A_{\mathcal{D}}, \alpha) \to (B_{\mathcal{D}}, \beta)$ *of* $\underline{\mathcal{C}}$ *determined by data as in [FrdI], Theorem 5.2, (i), (a), (b), (c), (d), is mapped to the morphism* $(A_{\mathcal{D}}, T_A, \tau_A) \to (B_{\mathcal{D}}, T_B, \tau_B)$ *of* $\underline{\mathcal{C}}^{\mathrm{tor}}$ *for which the data (a), (b) [as in the definition of* $\underline{\mathcal{C}}^{\mathrm{tor}}$*] is determined [in the evident way] by the data of [FrdI], Theorem 5.2, (i), (a), (b), and the data (c) [as in the definition of* $\underline{\mathcal{C}}^{\mathrm{tor}}$*] is the isomorphism of trivial* $\mathbb{B}(A_{\mathcal{D}})$*-torsors* $T_A^{\otimes d} \xrightarrow{\sim} \phi^* T_B$ *determined by multiplication by the element* $u_\phi \in \mathbb{B}(A_{\mathcal{D}})$ *of [FrdI], Theorem 5.2, (i), (d).*

<u>Question 24</u>: In view of Q23, might it not be possible to replace the current definition of a model Frobenioid by some sort of alternative definition which uses metrized line bundles or some variant of metrized line bundles as in [FrdII], Example 3.3, and would then include both this example and the examples referred to in Q23 as special cases? One could then discuss the relationship between this alternative definition and the current definition of a model Frobenioid.
<u>Response 24</u>:
As discussed in R22, the category $\mathcal{C}_0$ of [FrdII], Example 3.3, is *not* equivalent to a model Frobenioid. In fact, the *original definition* of a Frobenioid given in [FrdI], Definition 1.3, was intended/arrived at *precisely as an approach* to giving a *unified treatment* of the examples that you refer to (i.e., [FrdI], Examples 6.1 and 6.3; [FrdII], Examples 1.1 and 3.3), together with the *tempered Frobenioids* treated in [EtTh]. The *price* of insisting on such a unified approach may be seen in the *numerous technicalities* that appear in the theory of [FrdI], [FrdII], e.g., involving objects that are *not necessarily isotropic* and morphisms that are *not necessarily co-angular*. I do not know at the present time of another approach to obtaining such a *unified formulation*. (It would be interesting, however, if you are able to find an alternative approach to obtaining such a unified formulation.) I suppose that, from the point of view of using model Frobenioids, perhaps the closest kind of mathematical object that appears in [FrdI], [FrdII] to the sort of alternative version of a model Frobenioid that you seem to have in mind may be seen in the *poly-Frobenioids* of [FrdII], §5, i.e., in the case where one takes the *global* and *nonarchimedean* portions of the poly-Frobenioid to be *model Frobenioids* and the *archimedean* portion of the poly-Frobenioid to be a Frobenioid as in [FrdII], Example 3.3 — cf., e.g., [FrdII], Example 5.6.

Question 25: To what extent could one use the theory of adèles and idèles to simplify the portion of the theory of Frobenioids, as it is currently formulated, that is used in the papers on IUTeich?
Response 25:
As explained in R3, the only types of Frobenioids that appear in IUTeich are the types discussed in (F1), (F2), (F3), and (F4). Of these, the Frobenioids of (F3) and (F4) may be replaced by very simple objects. Also, the Frobenioids of (F2) are rather elementary and may be treated to a substantial extent without invoking the general theory of Frobenioids. On the other hand, the theory developed in [EtTh], §3, §4, §5, concerning the *tempered Frobenioids* of (F1) is rather nontrivial. I do not see any reason to believe that this theory may be simplified by using adèles/idèles. Here, let us recall that *model Frobenioids* (i.e., such as tempered Frobenioids) essentially amount to the data consisting of a *divisor monoid* and a *rational function monoid* (as in [FrdI], Theorem 5.2). This data is, in fact, *already in a form that is very close, in spirit, to objects that occur in the theory of idèles.* On the other hand, one *fundamental difference* between the conventional theory of idèles and the theory of Frobenioids is that the former starts from a *fixed number field* and concerns the study of various properties of idèles and related objects in a fashion that makes *essential use* of the way in which such objects were originally constructed from a number field. By contrast, the theory of Frobenioids starts from *abstract "combinatorial objects"* (such as *abstract monoids*) that are treated *without reference* to any rings/schemes (such as number fields). That is to say, the goal of the theory of Frobenioids is to see just how much of the structure that might arise, in various conventional situations, from some sort of ring or scheme, may in fact be *reconstructed purely category-theoretically*, i.e., *without reference* to any rings/schemes (cf. [FrdI], §I2). I do not see any reason to expect that the use of adèles/idèles would result in a meaningful simplification of the theory of such reconstruction algorithms.